Lecture 10

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1 Meaning of linear dependence and independence

On the last lecture we stated the result that if the system of vectors is linearly dependent, then at least one vector of the. system can be expressed as a linear combination of others. We gave an example how to do it. Now we'll give an example when it is not possible to express any vector as a linear combination of others.

Example 1.1. *Let*

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Here u_1, u_2 and u_3 are linearly independent and none of these vectors can be expressed as a linear combination of other 2 vectors. For example, for u_1 there are no real a and b such that

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = au_2 + bu_3 = a \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

2 Spanning sets

Definition 2.1. *Let V be a vector space. Vectors $v_1, v_2, \ldots, v_n$ are called a* ***spanning set*** *of V if every element of V is a linear combination of $v_1, v_2, \ldots, v_n$. In this case the space V is called a* ***span*** *of these vectors and it is denoted by $V = \langle v_1, v_2, \ldots, v_n \rangle$*

Example 2.2. *Consider the vector space $\mathbb{R}^3$. Then vectors $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, and $v_3 = (0,0,1)$ form a spanning set of $\mathbb{R}^3$, since if $u \in V$ equals to (a,b,c), then $u = av_1 + bv_2 + cv_3$.*

Example 2.3. *Consider the vector space $\mathbb{R}^3$. Then vectors $v_1 = (1,1,1)$, $v_2 = (1,1,0)$, and $v_3 = (1,0,0)$ form a spanning set of $\mathbb{R}^3$, since if $u \in V$ equals to (a,b,c), then $u = cv_1 + (b-c)v_2 + (a-b)v_3$. For example, $(4,6,1) = 1(1,1,1) + 5(1,1,0) - 2(1,0,0)$.*

Example 2.4. *Consider the vector space $P(t)$. Then vectors $1, t, t^2, t^3, \dots$ are a spanning set of $P(t)$ since it is clear that every polynomial can be expressed as a linear combination of these vectors.*

Example 2.5. *Consider a vector space $M_{2,2}$ of 2×2-matrices. Then the following matrices form a spanning set for $M_{2,2}$: $v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, since any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = av_1 + bv_2 + cv_3 + dv_4$.*

3 Homogeneous systems

As we saw in the previous lecture, in order to figure out whether the given vectors are linearly dependent or independent, we need to solve linear system with zeros in the right hand side. This leads us to the following topic.

Definition 3.1. *A system of linear equations is called **homogeneous** if right-hand sides of all its equations are equal to 0.*

Example 3.2. *The following system is homogeneous:*

$$\begin{cases} x_1 + 2x_2 + x_4 = 0 \\ 3x_1 + x_2 - x_3 + 5x_4 = 0 \end{cases}$$

The most important fact about homogeneous systems is that it has at least one solution, i.e. zero-solution — the solution where all the variables are equal to 0. As we saw before, for general linear systems there are 3 possible cases — no solutions, 1 solution, and infinitely many solutions. Here for homogeneous systems the first case may not happen. So, we have the following lemma:

Lemma 3.3. *The homogeneous system of linear equations may have unique solution or infinitely many solutions.*

Moreover, we can give another statement which tells us when the system has infinitely many solutions.

Theorem 3.4. *A homogeneous system with the number of equations less then the number of variables has infinitely many solutions.*

Proof. Let's transpose the system to a row echelon form. We will not get an equation of the form $0x_1 + \cdot + 0x_n = b$, where $b > 0$. Moreover, there will be free variables (since all the variables can not be leading — each equation has only one leading variable, so the number of leading variables is not greater than the number of equations, and the total number of variables is greater). So, the system will have infinitely many solutions, since we can assign any value to free variables. $\square$

Example 3.5. *The system from the example 3.2 has infinitely many solutions since the number of equations — 2 is less than the number of variables — 4.*

Example 3.6. *For this theorem to be true, it is important to consider only homogeneous systems: the following system has 4 variables and 2 equations as well, but has no solution:*

$$\begin{cases} x_1 + 2x_2 - x_3 + x_4 = 1 \\ 2x_1 + 4x_2 - 2x_3 + 2x_4 = 3 \end{cases}$$

since if we subtract the first equation multiplied by 2 from the second one, we'll get $0 = 1$*.*

4 Basis

On the last lecture we considered *spanning sets* of a vector space V — sets of vectors such that any vector from V can be expressed as a linear combination of vectors from this set.

Example 4.1. *Vectors*

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

form a spanning set for $\mathbb{R}^2$*. To check this one should take arbitrary vector and represent it as a linear combination of* u_1 *and* u_2*:*

$$\begin{pmatrix} a \\ b \end{pmatrix} = au_1 + bu_2 = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

do not form a spanning set for $\mathbb{R}^2$*. For example vector* $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *can not be represented as a linear combination of* v_1 *and* v_2*: there are no* $a, b \in \mathbb{R}$ *such that*

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Example 4.2. *Vectors*

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

form a spanning set for $\mathbb{R}^2$*. To check this one should take arbitrary vector and represent it as a linear combination of* u_1 *and* u_2*:*

$$\begin{pmatrix} a \\ b \end{pmatrix} = au_1 + bu_2 + 0u_3 = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Moreover, here we can give many different representations, e.g.

$$\begin{pmatrix} a \\ b \end{pmatrix} = au_1 + bu_2 + 0u_3 = (a-1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (b-1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For example,

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So, we see that there may be many different spanning sets for a vector space, and they may have different number of vectors. The spanning set in the example 4.1 is better than the spanning set in the example 4.2 since it contains less vectors.

Definition 4.3. *The system of vectors from vector space* V *is called* ***basis*** *if it is*

- *linearly independent and*
- *forms a spanning set for* V*.*

Example 4.4. *Consider the vector space* $\mathbb{R}^2$*. The system of vectors*

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is a basis. Let's check it.

- *This system forms a spanning set since any vector can be represented as a linear combination of* u_1*,* u_2 *and* u_3*:*

$$\begin{pmatrix} a \\ b \end{pmatrix} = au_1 + bu_2 = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- *This system is independent. Let's check it. We'll form a linear combination which is equal to 0, and see that it is trivial:*

$$x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  This is equivalent to the following system of linear equations:

$$\begin{cases} x \quad\quad = 0 \\ \quad\quad y = 0 \end{cases}$$

  So, we get that $x = 0,\ y = 0$. So, this linear combination is trivial, and thus the system of vectors is linearly independent.

Example 4.5. *Consider the vector space $\mathbb{R}^2$. The system of vectors*

$$u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is a basis. Let's check it.

- *This system forms a spanning set since any vector can be represented as a linear combination of u_1 and u_2:*

$$\begin{pmatrix} a \\ b \end{pmatrix} = au_1 + (b-a)u_2 = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (b-a) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- *This system is independent. Let's check it. We'll form a linear combination which is equal to 0, and see that it is trivial:*

$$x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  This is equivalent to the following system of linear equations:

$$\begin{cases} x \quad\quad\quad = 0 \\ x + y = 0 \end{cases}$$

  This system has only one solution $x = 0,\ y = 0$. So, this linear combination is trivial, and thus the system of vectors is linearly independent.

Example 4.6. *Consider a vector space $\mathbb{R}^2$. The system of vectors*

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

does not form a basis since it is not a spanning set for $\mathbb{R}^2$. For example we can never represent vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as a linear combination of u_1 and u_2.

Example 4.7. *Consider a vector space $\mathbb{R}^2$. The system of vectors*

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

does not form a basis since these vectors are not linearly independent. To check this let's form a linear combination and make it equal to 0:

$$x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is equivalent to the following system of linear equations:

$$\begin{cases} x \quad\quad + z = 0 \\ \quad\quad y + z = 0 \end{cases}$$

This system has a solution where not all variables are equal to 0, e.g. $x = 1,\ y = 1,\ z = -1$. *We don't need even to look for a solution, since this is a homogeneous system with two equations and three variables, and thus it has infinite number of solutions. So, these vectors are not linearly independent, since there is a nontrivial linear combination which is equal to 0-vector, e.g.*

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$